SIMPLICIAL LOCALIZATIONS AND HOW TO FIND THEM

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Abstract. Abstractly defining ∞-categorical localization is easy, but explicitly constructing it is hard. Following a series of papers by Dwyer and Kan, I describe the construction known as the *hammock localization* and use it to obtain a clearer picture of some important ∞-categories.

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This talk largely follows the work of Dwyer and Kan in [1], [2], and [3].

0. Conventions

Following Lurie, $\mathbf{Cat}_\Delta$ is the category of simplicially enriched categories. An object of $\mathbf{Cat}_\Delta$ will be called a *simplicial category*. Note that we have a natural inclusion $\mathbf{Cat}_\Delta \hookrightarrow \mathbf{sCat}$ by regarding a simplicially enriched category as a simplicial object in 1-categories having a discrete object set. Accordingly, I will write **sO-Cat** for the category of simplicial categories with object set O and morphisms being functors that are the identity on O. Similarly, I will implicitly assume that all simplicial graphs have discrete vertex set, and write **sO-Gr** for the category of simplicial graphs on the set O.

A *simplicial model category* is a closed module over the monoidal model category $(\mathbf{sSet}, \times, QK)$. That is, it is a simplicial category $\mathcal{C}$ tensored and cotensored over **sSet** such that we have a natural isomorphism $(K \times L) \otimes A \cong K \otimes (L \otimes A)$ and an adjunction of three variables $\mathrm{Hom}(K \otimes A, B) \cong \mathrm{Hom}(K, \mathrm{Hom}(A, B)) \cong \mathrm{Hom}(A, \mathrm{Hom}(K, B))$. Moreover, it is equipped with a model structure on $\mathcal{C}_0$ satisfying the pushout-product axiom: if f and g are cofibrations in **sSet** (with the Quillen-Kan model structure) and $\mathcal{C}_0$ respectively, then $f \otimes g$ is a cofibration in $\mathcal{C}_0$, which is trivial if f or g is. See Chapter II of [4] for more details.

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I will model ∞-categories using simplicial sets with the Joyal model structure and simplicial categories with the Bergner model structure. A weak equivalence in the Bergner model structure, also called a Dwyer-Kan equivalence, is a simplicial functor which is essentially surjective and induces a weak homotopy equivalence on mapping spaces. We have a Quillen equivalence $\mathbf{sSet} \underset{N}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{Cat}_\Delta$ where $\mathfrak{C}$ sends Δ^n to the simplicial category with objects $0, 1, \ldots, n$ such that $\mathrm{Hom}(i, j) = (\Delta^1)^{j-i-1}$ and composition is given by the inclusion of appropriate faces $(\Delta^1)^{\ell-i-1} \times (\Delta^1)^{j-\ell-1} \cong (\Delta^1)^{j-i-2} \hookrightarrow (\Delta^1)^{j-i-1}$. This uniquely determines the cocontinuous functor $\mathfrak{C}$ since $\mathbf{sSet}$ is a free cocompletion. The right adjoint N is given, as usual, by $N(\mathcal{C})_n = \mathrm{Hom}(\mathfrak{C}(\Delta^n), \mathcal{C})$ with the evident structure maps. The details can be found in section 1.2 of [5].

I'll try to be specific, but in general, "category" means 1-category and "∞-category" means quasicategory. Moreover, a *relative category* is a 1-category with a distinguished wide subcategory. If this subcategory satisfies the 2-out-of-3 rule, then we call the pair a *category with weak equivalences.*

1. Introduction

The 1-localization of a 1-category $\mathcal{C}$ at a class of morphisms W is defined by the universal property $\mathbf{Fun}(\mathcal{C}[W^{-1}]_1, \mathcal{D}) \cong \mathbf{Fun}^W(\mathcal{C}, \mathcal{D})$, the latter denoting the full subcategory of functors sending W to isomorphisms in $\mathcal{D}$. Symbolically, these are the functors sending W into the *core* of $\mathcal{D}$, $\mathcal{D}^{\simeq}$. Computing $\mathcal{C}[W^{-1}]_1$ is hard but doable, at least in principle: the objects are those of $\mathcal{C}$, and morphisms are reduced words in the elements of $\mathrm{Mor}(\mathcal{C}) \cup W^{-1}$.

The localization of an ∞-category $\mathcal{C}$ at a class of morphisms W is defined similarly by the universal property $\mathbf{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \simeq \mathbf{Fun}^W(\mathcal{C}, \mathcal{D})$.

Proposition 1.1. *The localization exists.*

Proof. I work in $\mathbf{sSet}$. Without loss of generality, W may be taken to be a subcategory (just take the subcategory generated by its morphisms).

First, assume $W = \mathcal{C}$. Then the universal property becomes $\mathbf{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \simeq \mathbf{Fun}(\mathcal{C}, \mathcal{D}^{\simeq})$. This property is satisfied by a fibrant replacement for $\mathcal{C}$ in the Quillen-Kan model structure, so it exists.

More generally, for an arbitrary subcategory W, we can form the (homotopy) pushout

$$\begin{array}{ccc} W & \longrightarrow & W[W^{-1}] \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}[W^{-1}] \end{array}$$

and one easily verifies that a Joyal-fibrant replacement for this satisfies the desired universal property. □

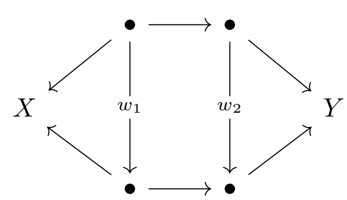
So we know that $\mathcal{C}[W^{-1}]$ exists, but this doesn't give us any way to directly compute it (other than with the small object argument, which is clearly unfeasible). In particular, it is far from obvious how to express the ∞-localization of a relative category $(\mathcal{C}, W)$ in terms of its morphisms. This is a problem, because many of our favorite ∞-categories are defined to be the localization of model categories. Examples include $\mathbf{Cat}_\infty$, $\mathbf{Spaces}$, $\mathbf{Ch}^+(\mathcal{A})$, and $\mathbf{Sp(Top)}$.

To fix this, we will describe an explicit localization procedure passing through $\mathbf{Cat}_\Delta$. This procedure will generalize the "reduced word" construction for 1-localization, replacing words with the more general "hammocks". As a reward for our work, we will find very simple descriptions for some of these categories.

2. The Hammock Localization

Let $(\mathcal{C}, W)$ be a relative category. Given a word $\mathfrak{m}$ in the two letters $\mathcal{C}$ and W^{-1}, we obtain an associated class of prospective pasting diagrams. For example, a diagram for the word $\mathcal{C}\mathcal{C}W^{-1}$ looks like $X \xleftarrow{w} \bullet \xrightarrow{c_1} \bullet \xrightarrow{c_2} Y$. (Note that the empty word is permitted.)

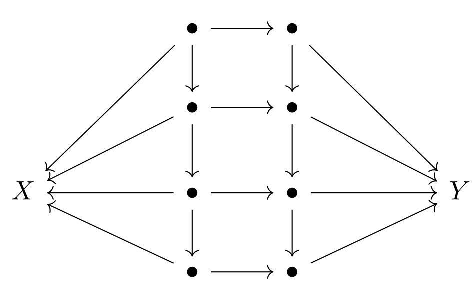
We define a morphism between such diagrams to be a commutative diagram whose vertical maps are in W and whose left- and rightmost maps are the identity, e.g. the following.



We thus require these diagrams to have the same word and the same leftmost and rightmost objects.

Definition 2.1. This is the *hammock category* for (X, Y) in the word $\mathfrak{m}$, denoted $N^{-1}\mathfrak{m}(C, Y)$. A k-simplex of its nerve $\mathfrak{m}(X, Y)$ is called a *hammock* of width k, length$|\mathfrak{m}|$, and type $\mathfrak{m}$.

Example 2.2. A hammock of width 3 and type $\mathcal{C}\mathcal{C}W^{-1}$. The appearance of these diagrams gives them their name.



We can put together all of these nerves as X and Y vary to get a simplicial graph with vertex set $\mathrm{Ob}(\mathcal{C})$, which is also called $\mathfrak{m}$.

As in the 1-categorical case, we want to reduce our diagrams in order to avoid redundancy in our localization.

Definition 2.3. A hammock is called *reduced* if

(i) Its type $\mathfrak{m}$ has no adjacent repeated letters, and
(ii) No column of horizontal morphisms consists entirely of identity morphisms.

Any hammock can be put into reduced form by removing identity columns and composing adjacent $\mathcal{C}$ columns and W^{-1} columns. The reduced form can thus be viewed as a collection of representatives for equivalence classes of the evident relation. Now we have the terminology to define our localization.

Definition 2.4. The *hammock localization* of a relative 1-category $(\mathcal{C}, W)$ is the simplicial category $L^H\mathcal{C}$ with the same object set as $\mathcal{C}$, having hom objects $L^H C(X,Y) = (\bigcup_{\text{words } \mathfrak{m}} \mathfrak{m}(X,Y)) \,/ \sim$, where $\sim$ is the relation described above. Composition is given by concatenation.

We can view a k-simplex of $L^H C(X,Y)$ as a reduced hammock of width k, in which case the composition operation becomes concatenation followed by reduction. This construction gives us a functor $L^H : \mathbf{RelCat} \to \mathbf{Cat}_\Delta$. In fact, since it leaves the object set unchanged, it restricts to a functor from **RelO-Cat** to **sO-Cat**, where O is any fixed object set.

We have our construction, but how do we know it is actually a localization in the sense previously defined? To see this, we use the so-called "standard simplicial localization" L described by Dwyer and Kan in [1]. The construction is as follows. Fixing an object set O, we have a free-forgetful adjunction between **O-Cat** and **O-Gr**. The comonad of this adjunction gives us a simplicial bar construction $F_* : \mathbf{O\text{-}Cat} \to \mathbf{sO\text{-}Cat}$, which gives us a simplicial resolution of a category by free categories. Then it is not hard to show that the canonical map $\varphi : F_*\mathcal{C} \to \mathcal{C}$ is a weak equivalence of simplicial categories. As is often the case with resolutions, this should be thought of as a cofibrant replacement; in fact, it defines a functorial cofibrant replacement for $(\mathcal{C}, W)$ in the Bergner model structure on $\mathbf{RelCat}_\Delta$.[1] Now we define our localization functor as follows.

Definition 2.5. The *standard simplicial localization* $L\mathcal{C}$ of a relative 1-category $(\mathcal{C}, W)$ is the dimensionwise localization $F_*C[F_*W^{-1}]$.

The primary result of [1] is that dimensionwise localization yields a localization functor on the category of relative simplicial categories which behaves correctly on cofibrant objects. In particular:

(i) If (V, V) is a cofibrant relative simplicial category, then LV is an ∞-groupoid with the correct homotopy type;
(ii) The pushout used to define the localization in general is a homotopy pushout in the cofibrant case; and
(iii) Localization preserves weak equivalences.

This is enough to show that L, or more accurately its left derived functor, is the correct localization.

I'll skip the proof, but suffice to say it mainly involves two useful tricks: reducing to the free case (this is where cofibrancy is relevant) and using the homotopy-invariance of the diagonal of a bisimplicial set. Similar arguments, together with a two-sided Grothendieck construction for simplicial categories (which is applied to $N^{-1}\mathfrak{m}$), yield the following result.

Proposition 2.6. *The natural maps* $L^H\mathcal{C} \leftarrow \operatorname{diag} L^H F_*\mathcal{C} \to F_*\mathcal{C}[F_*W^{-1}] = L\mathcal{C}$ *are weak equivalences of simplicial categories.*

The proof uses an alternate construction of $L^H\mathcal{C}$, which can sometimes be useful. Define Π to be the category whose objects are partitions (S, T) of sets of the form

[1] A cofibration in $\mathbf{Cat}_\Delta$ is a retract of the inclusion map into the coproduct with a free simplicial category; a cofibration in $\mathbf{RelCat}_\Delta$ is just a cofibration between cofibrant categories.

$\{1, 2, \ldots, n\}$, where n is any nonnegative integer, and whose morphisms are functions which preserve both the order (weakly) and the partition. We can view this as a category of words in two letters, where S corresponds to $\mathcal{C}$ and T corresponds to W^{-1}. Accordingly, we have a functor $\lambda\mathcal{C} : \Pi \to$ **sO-Gr** which sends a word $\mathfrak{m}$ to the simplicial graph $\mathfrak{m}$ described above. We have a natural reduction map from each such graph to the underlying simplicial graph of $L^H\mathcal{C}$, and it is shown in [1] that these assemble into an isomorphism $\operatorname{colim} \lambda\mathcal{C} \cong L^H\mathcal{C}$.

This proposition, together with what we know about the standard simplicial localization, tells us that L^H is truly a localization in the usual sense.

3. Homotopy Calculi of Fractions

A hammock gives us an explicit representation of a k-cell in a localization, but it can be very big. If our goal is to find a workable hands-on representation of a localization, we need some sort of reduction. Fortunately, one is available.

Definition 3.1. A relative category $(\mathcal{C}, W)$ is said to admit a *homotopy calculus of (two-sided) fractions* if the natural maps $W^{-1}C^{i+j}W^{-1} \to W^{-1}C^iW^{-1}C^jW^{-1}$ and $W^{-1}W^{i+j}W^{-1} \to W^{-1}W^iW^{-1}W^jW^{-1}$ are weak equivalences of simplicial graphs for all $i, j \in \mathbb{Z}^+$.

There are analogous concepts of homotopy calculi of left fractions and right fractions. (The direction refers to the side W^{-1} is on, so e.g. a homotopy calculus of left fractions deals with a map out of $W^{-1}C^{i+j}$.) As it turns out, these are stronger than the two-sided calculus described above, but any of them can be used to give a greatly simplified representation of the hammock localization.

Theorem 3.2. *Suppose the relative category $(\mathcal{C}, W)$ admits a homotopy calculus of fractions. Then the reduction map $W^{-1}CW^{-1} \to L^HC$ is a weak equivalence of simplicial graphs. Analogous results hold for homotopy calculi of left and right fractions.*

Proof. The idea of the proof is to use the characterization of L^HC as a colimit to build it out of word graphs of the form appearing in our homotopy calculi, and use the provided weak equivalences to reduce down to the smallest possible word. The details involve representing the relevant operations as composition with certain endofunctors of Π and verifying that several natural transformations between them are weak equivalences, which is done using a combination of explicit computation and the two-sided Grothendieck construction mentioned above. $\square$

There are several situations in which these homotopy calculi can be guaranteed to exist. For example, if $(\mathcal{C}, W)$ admits a *classical* calculus of left fractions (a condition that allows weak equivalences to be "shifted around" and "cancelled" in a certain one-sided way), then it also admits a homotopy calculus of left fractions (and dually for right fractions). This is in some sense a trivial case, however, because the existence of these classical calculi guarantees that the localization is equivalent to a 1-category. More interesting is the following result:

Proposition 3.3. *Let $(\mathcal{C}, W)$ be a category with weak equivalences. Suppose that any span diagram with one of the legs in W can be naturally completed to a commutative square where each new arrow is in W if the parallel original arrow is. Then $(\mathcal{C}, W)$ admits a homotopy calculus of left fractions.*

In particular, if $(\mathcal{C}, W)$ is a category with weak equivalences such that W is closed under pushouts, then it admits a homotopy calculus of left fractions.

There is also a two-sided version of this proposition, and its hypotheses should seem very familiar.

Proposition 3.4. *Let $(\mathcal{C}, W)$ be a category with weak equivalences such that there are wide subcategories W_1 and W_2 of W satisfying the following conditions:*

(i) *Any span diagram as shown can be naturally completed to a commutative square with $v \in W_1$, and moreover g is in W if f is;*

$$\begin{array}{ccc} X & \xrightarrow{u \in W_1} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \underset{v}{\dashrightarrow} & Y' \end{array}$$

(ii) *Any cospan diagram as shown can be naturally completed to a commutative square with $u \in W_2$, and moreover f is in W if g is; and*

$$\begin{array}{ccc} X' & \overset{u}{\dashrightarrow} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow[v \in W_2]{} & Y \end{array}$$

(iii) *Each morphism $w \in W$ admits a functorial factorization $w = w_2 w_1$ with $w_1 \in W_1$ and $w_2 \in W_2$.*

Then $(\mathcal{C}, W)$ admits a homotopy calculus of fractions.

This hypotheses of this proposition are extraordinarily similar to the properties of a model category! One need only take W_1 and W_2 to be acyclic cofibrations and acyclic fibrations respectively, and assume that the model category in question has functorial factorization. Thus, we have the following nice reduction for model categories.

Corollary 3.5. *Every Thomason model category admits a homotopy calculus of fractions.*

Just like how the structure of a model category gives us a simple way to represent morphisms in the 1-localization, the structure of a Thomason model category gives us a simple way to represent cells of arbitrary dimension in a model for the ∞-localization. This is why we can think of model categories as "presentations" for ∞-categories. In fact, the result holds without the assumption of functorial factorization, but the proof is more difficult. There is one more special case, however, which includes some of our favorite model categories.

4. Simplicial Model Categories

The following result is incredibly powerful, as it allows us to compute the localization of a simplicial model category directly.

Theorem 4.1. *Let M be a simplicial model category. Then $L^H M_0$ and M are naturally equivalent as simplicial categories.*

Examples 4.2.

(i) **Spaces** is defined to be the localization of **sSet** with the Quillen-Kan model structure. Since this is obviously a simplicial model category, we find that the usual self-enrichment of **sSet** coincides with the ∞-categorical simplicial hom object.

(ii) $\mathbf{Cat}_\infty$ is typically defined to be the localization of **sSet** with the Joyal model structure, which is most certainly *not* a simplicial category. However, it also admits a presentation by the model category of marked simplicial sets $\mathbf{sSet}_+$, which is a simplicial category. (In fact, the inclusion of **sSet** as simplicial sets with no marked edges induces an equivalence on localizations.) Since the underlying simplicial enrichment of $\mathbf{sSet}_+$ is given by the core of the simplicial hom, we find that **qCat**, the simplicial category of quasicategories and cores of their mapping spaces, has coherent nerve (Joyal equivalent to) $\mathbf{Cat}_\infty$.

(iii) The category of simplicial spectra with the Bousfield-Frielander model structure is a simplicial model category, from which we can compute the simplicial hom spaces of **Spectra**.

(iv) The model category $\mathbf{Ch}(\mathcal{A})$ is isomorphic to **sAb** with the simplicial model structure induced from enrichment, hence the coherent nerve of **sAb** is equivalent to the stable ∞-category of positively graded chain complexes.

The proof of this proposition is via a series of lemmata involving (co)simplicial resolutions.

Definition 4.3. A *simplicial resolution* of an object A in a model category is a Reedy-fibrant simplicial object Y_* together with a weak equivalence $A \to Y_0$. If the weak equivalence is a cofibration and Y_* is also Reedy cofibrant, we say this is a *special* simplicial resolution. Ordinary and special cosimplicial resolutions X^* of an object B are defined dually.

Lemma 4.4. *Let M be a model category, and let X and Y be objects of M. Then X has a special cosimplicial resolution X^*, Y has a special simplicial resolution Y_*, and for any (possibly non-special) (co)simplicial resolutions of these objects, there is a natural weak equivalence between* $\operatorname{diag} M(X^*, Y_*)$ *and $L^H M(X,Y)$. Moreover, if X is cofibrant, they are equivalent to $M(X, Y_*)$, and if Y is fibrant, they are equivalent to $M(X^*, Y)$.*

Proof. One may assume that the resolutions are special. Moreover, since M admits a homotopy calculus of fractions, we have a weak equivalence $(W^c)^{-1}M(W^f)^{-1}(X,Y) \to L^H M(X,Y)$. But this first object and the diagonal can both be expressed as homotopy colimits, and their diagrams are related by a functor which can be shown to be cofinal. The last part follows because we can evaluate the diagonal by taking the homotopy colimit in the first argument or the homotopy limit in the second. □

This can be used to show the following corollary.

Lemma 4.5. *If M is a simplicial model category, then for each $k \geq 0$, the iterated degeneracy map $L^H M_0 \to L^H M_k$ is a weak equivalence.*

This is proven by using (co)simplicial resolutions together with the previous result to relate the simplicial structure to the hammock localization.

Proof. The theorem now follows by a diagonal argument. To be precise, one uses the fact that a homotopy colimit over a simplicial diagram in simplicial sets can have the simplicial factors switched without altering the homotopy colimit, as both are equal to the diagonal of the associated bisimplicial set. In this case, one can evaluate the diagonal of $L^H M$ either over the simplicial structure of the category (which is homotopically constant and thus yields $L^H M_0$) or over the hammock localization (which yields M by our first lemma). □

5. Bibliography

References

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